Indian Statistical Institute
First Semester Examination 2005-2006
B.Math II Year
Differential Equations

Time: 3 hrs Date: 24-11-05 [Max marks :]

1. It is known that $f(x) = e^{\alpha x}$ is a solution of

$$\left(\frac{d}{dx} - \alpha\right)^3 f(x) = 0$$

Show that any solution of the above equation is $p(x)f(x)$ for a suitable polynomial of degrees ≤ 2. (Hint: for any nice function $q\left(\frac{d}{dx} - d\right)[qf] = q'f$. [3]

2. Let f be a spherically symmetric function on R^n. Show that

$$\sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} f(r) = f''(r) + \frac{n-1}{r} f'(r)$$

where $r = (x_1^2 + x_2^2 + \ldots + x_n^2)^{1/2}$ [3]

3. Let $p_n(x) = \frac{1}{2^n L^n} \frac{d^n}{dx^n}(x^2-1)^n$ for $n = 0, 1, 2, 3, \ldots$ Calculate $\int\limits_{-1}^{1} dx[p_n(x)]^2$ using $\int\limits_{-1}^{1} dx p_m(x) p_k(x) = 0$ for $m \neq k$

4. Let $f : R^n \to \mathbb{C}$ be any continuous function of compact support for $n = 3, 4, 5, \ldots$

Let $g_\epsilon(x) = (x^2 + \epsilon^2)^{(2-n)/2}$. Define $u_\epsilon(x) = \int_{R^n} dy f(y) g_\epsilon(x-y)$

(a) Show that $(\Delta g_\epsilon)(x) = (2-n) \cdot n\epsilon^2 [x^2 + \epsilon^2]^{-(n+2)/2}$ [4]

Hint: You can use question no 2.

(b) Let $h(x) = (2-n)n \cdot (x^2+1)^{-(n+2)/2}$, show that $\int dx \ \ |h(x)| < \infty$. Put $k_n = \int dx \ \ h(x)$. Hint: Use $dx = r^{n-1} dr dw$, the change of variable, where dw is the surface measure on $S^{n-1} = \{x \in R^n := ||x|| = 1$. [2]

(c) Show that ΔU_ϵ converges to $k_n f$ uniformly on compact subsets of R^n. [4]

5. Let P, Q have convergent power series given by $P(x) = p_0 + p_1 x + p_2 x^2 + \ldots,$

$Q(x) = q_0 + q_1 x + q_2 x^2, + \ldots$

Show that the equation

$$u''(x) + P(x)u'(x) + Q(x)u(x) = 0$$

has, for each a_0, a_a, a convergent power series solution with $u(0) = a_0$ and $u'(0) = a_1$ [6]

6. (a) Let $T : (X, d) \to (X, d)$ be a continuous function on a complete metric space. Assume that $S = T^p = T(T(\ldots T))\cdots)$[$p$ times] satisfies $d(Sx,\ Sy) \leq kd(x, y)$ for some $k < 1$. Show that T has a unique fixed point. [1]

   (b) Let $a \leq x_0 \leq b$. Show that

$$\int\limits_{t_0}^{u} dt_1 \int\limits_{t_0}^{t_1} dt_2 \int\limits_{t_0}^{t_2} dt_3 \ldots \int_{t_0}^{t_{k-1}} dt_k = \frac{(u - t_0)^k}{(k)} \qquad \text{for } \ a \leq t \leq b$$

[2]

   (c) Let $f : [a, b] \times R \to R$ be a continuous function such that there exists L such that

$$|f(t, x) - f(t, y)| \leq L|x - y|$$

   for all x, y in R, all t in $[a, b]$. [4]

   Show that the equation

$$u'(t) = f(t, u(t)),\ u(t_0) = x_0$$

   has a unique solution on $[a, b]$.

7. Let $g_0, g_1, g_2 : [a, b] \to R$ be continuous functions. Let f_1, f_2, f_3 be solutions of the third order linear equation

$$u^{(3)}(t) + g_2(t)u^{(2)}(t) + g_1(t)u^{(1)}(t) + g_0(t)u(t) = 0$$

   Let $M(t) = \begin{bmatrix} f_1(t) & f_2(t) & f_3(t) \\ f_1'(t) & f_2'(t) & f_3'(t) \\ f_1''(t) & f_2''(t) & f_3''(t) \end{bmatrix}$

   and $W(t) = \det M(t)$

   (a) find a differential equation involving W^1, W, g_0, g_1, g_2 and solve it. [3]

   (b) If row rank $M(t_0) = 3$ for some t_0 then show that column rank $M(s) = 3$ for all s. [2]

8. Let $f : [0, \infty] \to \mathbb{C}$ be any bounded continuous function. Show that every solution of the second order equation

$$u'' + 3u' + u = f$$

   is bounded on $[0, \infty)$. [6]

9. Let $p(t,x) = (4\pi t)^{-n/2} e^{-x^2/(4t)}$ $t > 0$ and x in R^n. Let $f : r^n \to \mathbb{C}$ be any bounded continuous function. Then (a) $\frac{\partial p}{\partial t} = \sum\limits_1^n \frac{\partial^2 p}{\partial x_j^2}$ [2]

(b) For $u(0,x) = \int dy f(x-y) p(t,y)$ show that $u(0,x) \to f(x)$ as $t \to 0$.

10. Show that the gamma function Γ defined on $(0,\infty)$ by

$$\Gamma(x) = \int\limits_0^\infty dt e^{-t} t^{x-1}$$

is convex on $(0,\infty)$. [5]